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Proof of the Fukui conjecture via resolution of singularities and related methods. V

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Abstract The present article is a direct continuation of part IV of this series. The Local Analyticity Proposition (LAP1), which admits a proof via resolution of singularities is a major key to proving the Fukui conjecture via resolution of singularities and related methods. By LAP1, the essential part of the mechanism of the "asymptotic linearity phenomena" is extracted and is elucidated by using tools from the theory of algebraic and analytic curves. Here in the present article, we complete the proof of the LAP1 by using fundamental tools developed in parts III and IV of this series, thus completing the proof of the Fukui conjecture via resolution of singularities and related methods. This series of articles I-V establishes, for the first time, a new

Dedicated to the memory of Prof. Kenichi Fukui (1918–1998).

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linkage between (i) the mathematical field of resolution of singularities and (ii) the chemical field of additivity problems tackled and solved in a unifying manner via the repeat space theory (RST), which is the central theory in the First and Second Generation Fukui Project. A new development called the Matrix Art Program in the Second Generation Fukui Project has also been expounded with a graphical representation of energy band curves of a carbon nanotube.

Keywords Fukui conjecture · Repeat space theory (RST) · Asymptotic linearity theorem (ALT) · Resolution of singularities · Matrix art · Asymptotic linearity theorem extension conjecture (ALTEC)

1 Introduction

This article is a direct continuation of the previous part IV [1] of this series of articles, which have been devoted to cultivating a new interdisciplinary region between chemistry and mathematics along the unifying spirit of the First and the Second Generation Fukui Project (cf. [1–15] and references therein).

The Local Analyticity Proposition (LAP1), which is also referred to as the Target Proposition in parts II, III, and IV of this series of articles [1–3], is a major key to proving the Fukui conjecture via resolution of singularities and related methods.

In part II of this series [3], we have established the following sequence of logical implications:

$$\text{LAP1} \Rightarrow \text{Functional ALT} \Rightarrow \text{the Fukui conjecture},$$

where the ALT stands for the Asymptotic Linearity Theorem (cf. [3]).

Here in this article, we prove the LAP1 (Target Proposition) using Proposition 3.5.C (Intermediate Target Proposition), which was established in part IV of this series [1]. The assertion of the Intermediate Target Proposition involves graphs $\Gamma(h_i \circ \nu_{i,j})$ of complex-valued functions $h_i \circ \nu_{i,j}$ defined on discs on the complex plane. On the other hand the assertion of the Target Proposition involves graphs $\Gamma(h_i)$ of real-valued functions h_i defined on intervals on the real line. We remark that energy band curves given in ref. [8] are locally expressed in terms of the graphs $\Gamma(h_i)$ of real-valued functions h_i defined on intervals on the real line.

The crucial step of establishing the Target Proposition in Sect. 2 via the Intermediate Target Proposition lies in the topological and complex analysis of the structures of $\bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \Gamma(h_i \circ \nu_{i,j})$ and $\bigcup_{i=1}^{n} \Gamma(h_i)$, which were called multi-function-sets in previous parts III and IV of this series. The reader is referred to a series of papers [8,9] that analyze energy band curves of carbon nanotubes, which helps provide concrete examples of how the above mentioned multi-function-sets $\bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} \Gamma(h_i \circ \nu_{i,j})$ and $\bigcup_{i=1}^{n} \Gamma(h_i)$ are analyzed in the present paper so as to reach our goal of proving the Target Proposition in this article.

In Sect. 3, a new development called the Matrix Art Program in the Second Generation Fukui Project has been expounded with a graphical representation of energy band curves of a carbon nanotube. It is noteworthy that Matrix Art which involves

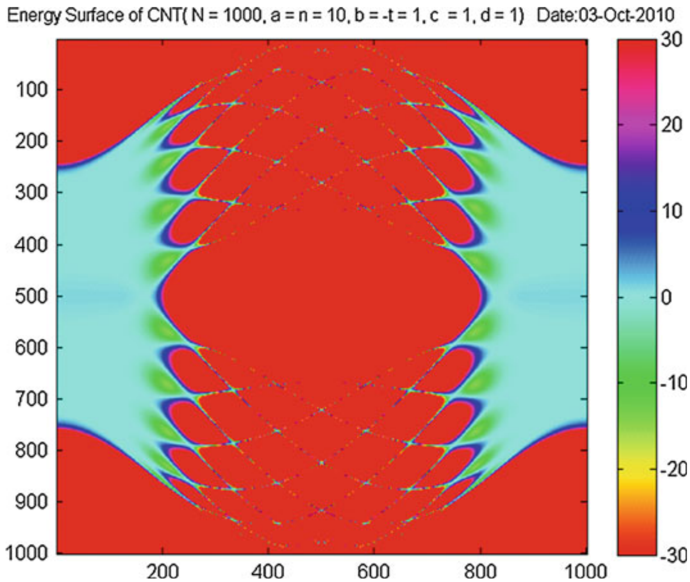

Fig. 1 Matrix art of the (θ, λ) energy surface and energy band curves of a carbon nanotube

fractal structure with self-similarity (see e.g. Fig. 1 in ref. [5] and Fig. 2 in the present paper) has been utilized to prove the **Asymptotic Linearity Theorem Extension Conjecture** (ALTEC), which was presented in article [21] (entitled "Open problem, Magic Mountain and Devil's Staircase swapping problems") and is important in view of the Fukui conjecture—a guiding conjecture in the repeat space theory (RST).

2 Proof of the target proposition

Throughout, we retain the notation employed in the preceding parts I–IV of this series of articles [1–4]. (The reader is referred to refs. [1–4] for the notation). Let $\mathbb{Z}^+, \mathbb{Z}_0^+, \mathbb{Z}, \mathbb{R}^+, \mathbb{R}_0^+, \mathbb{R}$, and $\mathbb{C}$ denote respectively, the set of all positive integers, nonnegative integers, integers, positive real numbers, nonnegative real numbers, real numbers, complex numbers.

This section is devoted to proving our Target Proposition. For this purpose, we need some preparation.

Notation 2.1 Let S_1 and S_2 be nonempty sets such that $S_1 \subset S_2$, then $\mathrm{inc}(S_1, S_2)$ denotes the inclusion map of S_1 into S_2.

Proposition 2.1 *Let $a, b \in \mathbb{R}$ with $a < b$ and let $I = [a,\ b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^+$ given by*

$$p = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q. \tag{2.1}$$

Define f: $I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta), \tag{2.2}$$

and define $g : I \times \mathbb{C} \to \mathbb{C}$ by

$$g(\theta, \lambda) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta). \tag{2.3}$$

Then, we have

(i) $f^{-1}(0) = g^{-1}(0) \cap (I \times \mathbb{R})$.
Moreover, there exist an open connected set $\hat{I}$ of $\mathbb{C}$ with $\hat{I} \supset I$ and an analytic function $\hat{f} : \hat{I} \times \mathbb{C} \to \mathbb{C}$ such that the following statements hold:
(ii) $g^{-1}(0) = \hat{f}^{-1}(0) \cap (I \times \mathbb{C})$.
(iii) $f^{-1}(0) = \hat{f}^{-1}(0) \cap (I \times \mathbb{R})$.

Proof Let $\xi \in I$. Since $c_1, \ldots, c_q \in C^\omega(I)$, for each $i \in \{1, \ldots, q\}$, there exists a power series $\psi_i^\xi \in \mathbb{R}\{z\}$ with the radius of convergence $\rho_i^\xi \in]0, \infty]$ such that

$$c_i(\theta) = \psi_i^\xi(\theta - \xi) \tag{2.4}$$

for all $\theta \in I$ with $|\theta - \xi| < \rho_i^\xi$. (Recall that for $x \in \mathbb{C}$ and $r \in \mathbb{R}^+$, $\Delta_x(r) := \{y \in \mathbb{C} : |y - x| < r\}$.) Let

$$D_\xi := \bigcap_{1 \le i \le q} \Delta_\xi(\rho_i^\xi). \tag{2.5}$$

Then, for each $i \in \{1, \ldots, q\}$, $\hat{c}_i^\xi : D_\xi \to \mathbb{C}$ defined by

$$\hat{c}_i^\xi(\theta) := \psi_i^\xi(\theta - \xi) \tag{2.6}$$

is analytic on D_ξ, where we consider ψ_i^ξ as an element of $\mathbb{C}\{z\}$.

Next, note that the compact subset I of $\mathbb{C}$ is covered by the open subsets D_ξ of $\mathbb{C}$:

$$I \subset \bigcup_{\xi \in I} D_\xi, \tag{2.7}$$

thus I has a finite subcover. In other words, there exist $m \in \mathbb{Z}^+$ and $\xi_1, \ldots, \xi_m \in I$ such that

$$I \subset \bigcup_{1 \le j \le m} D_{\xi_j}. \tag{2.8}$$

If $m \geq 2$, by considering a rearrangement, we may and do assume that

$$\left(\bigcup_{1 \leq j \leq l} D_{\xi_j} \right) \cap D_{\xi_{l+1}} \neq \emptyset \tag{2.9}$$

for all $l \in \{1, \ldots, m-1\}$. Fix such an m and $\xi_1, \ldots, \xi_m$ as described above, and let

$$\hat{I} := \bigcup_{1 \leq j \leq m} D_{\xi_j}. \tag{2.10}$$

Relation (2.9) implies that $\bigcup_{1 \leq j \leq l} D_{\xi_j}$ is open and connected in $\mathbb{C}$ for all $l \in \{1, \ldots, m\}$. Bearing in mind this fact and the fact that each of $\hat{c}_1^{\xi_j}, \ldots, \hat{c}_q^{\xi_j}$ is analytic on its domain D_{ξ_j}, we easily see that for each $i \in \{1, \ldots, q\}$ there exists a unique analytic function $\hat{c}_i : \hat{I} \to \mathbb{C}$ so that

$$\hat{c}_i | D_{\xi_j} = \hat{c}_i^{\xi_j} \quad (\forall j \in \{1, \ldots, m\}). \tag{2.11}$$

For each $i \in \{1, \ldots, q\}$, define $\hat{c}_i : \hat{I} \to \mathbb{C}$ to be the analytic function that satisfies (2.11).

Notice now that

$$\hat{c}_i \circ \mathrm{inc}(I, \hat{I}) = \mathrm{inc}(\mathbb{R}, \mathbb{C}) \circ c_i \tag{2.12}$$

for all $i \in \{1, \ldots, q\}$.

Define $\hat{f} : \hat{I} \times \mathbb{C} \to \mathbb{C}$ by

$$\hat{f}(\theta, \lambda) = \lambda^q + \hat{c}_1(\theta)\lambda^{q-1} + \cdots + \hat{c}_q(\theta), \tag{2.13}$$

and note that $\hat{f}$ is analytic on its domain. Recalling the definitions of f and g, we then have

$$g \circ \mathrm{inc}(I \times \mathbb{R}, I \times \mathbb{C}) = \mathrm{inc}\,(\mathbb{R}, \mathbb{C}) \circ f, \tag{2.14}$$
$$\hat{f} \circ \mathrm{inc}(\hat{I} \times \mathbb{R}, I \times \mathbb{C}) = \mathrm{inc}(\mathbb{C}, \mathbb{C}) \circ g, \tag{2.15}$$
$$\hat{f} \circ \mathrm{inc}(\hat{I} \times \mathbb{R}, I \times \mathbb{C}) = \mathrm{inc}(\mathbb{R}, \mathbb{C}) \circ f, \tag{2.16}$$

i.e., the following diagram is commutative:

$$\begin{tikzcd}
I \times \mathbb{R} \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] & \mathbb{R} \arrow[d, "\mathrm{inc}"] \\
I \times \mathbb{C} \arrow[r, "g"] \arrow[d, "\mathrm{inc}"'] & \mathbb{C} \arrow[d, "\mathrm{inc}"] \\
\hat{I} \times \mathbb{C} \arrow[r, "\hat{f}"] & \mathbb{C}
\end{tikzcd}$$

By using this commutative diagram, one easily verifies that statements (i), (ii), and (iii) are true. This completes the proof. □

Proposition 2.2 *The notation and the assumptions being as in Proposition 2.1, the following statements are equivalent:*

(i) *For any $\theta \in I$, the polynomial*

$$\mathrm{Ev}_{\theta}\,(p) = \lambda^q + c_1(\theta)\lambda^{q-1} + \cdots + c_q(\theta) \tag{2.17}$$

over the field $\mathbb{R}$ has q real roots.

(ii) $f^{-1}(0) = g^{-1}(0)$.

(iii) $\hat{f}^{-1}(0) \cap (I \times (\mathbb{C} - \mathbb{R})) = \emptyset$.

Proof By the definition of g, (i) is equivalent to saying that

$$g^{-1}\,(0) \cap (I \times (\mathbb{C} - \mathbb{R})) = \emptyset, \tag{2.18}$$

that is

$$g^{-1}\,(0) = g^{-1}\,(0) \cap (I \times \mathbb{R}). \tag{2.19}$$

Thus, by the definition of f, (i) is equivalent to (ii):

$$g^{-1}\,(0) = f^{-1}\,(0)\,. \tag{2.20}$$

Statement (ii) is equivalent to statement (iii), since by Proposition 2.1(ii) and 2.1(iii), equality (2.20) is equivalent to

$$\hat{f}^{-1}(0) \cap (I \times \mathbb{C}) = \hat{f}^{-1}(0) \cap (I \times \mathbb{R}), \tag{2.21}$$

which is equivalent to (iii). □

We summarize the key facts established so far in a practical form, which will be used for proving Proposition 2.4 (Target Proposition).

Proposition 2.3 *Let a, $b \in \mathbb{R}$ with $a < b$ and let $I = [a, b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^{+}$ given by*

$$p = \lambda^{q} + c_{1}\lambda^{q-1} + \cdots + c_{q}. \tag{2.22}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_{\theta}(p) = \lambda^{q} + c_{1}(\theta)\lambda^{q-1} + \cdots + c_{q}(\theta) \tag{2.23}$$

over the field $\mathbb{R}$ has q real roots.

Define f: $I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^{q} + c_{1}(\theta)\lambda^{q-1} + \cdots + c_{q}(\theta). \tag{2.24}$$

Then, there exist an open connected set $\hat{I}$ of $\mathbb{C}$ with $\hat{I} \supset I$ and an analytic function $\hat{f} : \hat{I} \times \mathbb{C} \to \mathbb{C}$ such that

$$f^{-1}(0) = \hat{f}^{-1}(0) \cap (I \times \mathbb{C}) = \hat{f}^{-1}(0) \cap (I \times \mathbb{R}). \tag{2.25}$$

Proof The conclusion readily follows from Propositions 2.1 and 2.2. □

The rest of this section is devoted to proving the Target Proposition 2.4.

Proposition 2.4 (Target Proposition). *Let a, $b \in \mathbb{R}$ with $a < b$ and let $I = [a,\ b]$. Let $p \in C^{\omega}(I)[\lambda]$ be a monic polynomial of degree $q \in \mathbb{Z}^{+}$ given by*

$$p = \lambda^{q} + c_{1}\lambda^{q-1} + \cdots + c_{q}. \tag{2.26}$$

Suppose that for any $\theta \in I$, the polynomial

$$\mathrm{Ev}_{\theta}\ (p) = \lambda^{q} + c_{1}(\theta)\lambda^{q-1} + \cdots + c_{q}(\theta) \tag{2.27}$$

over the field $\mathbb{R}$ has q real roots.

Define f: $I \times \mathbb{R} \to \mathbb{R}$ by

$$f(\theta, \lambda) = \lambda^{q} + c_{1}(\theta)\lambda^{q-1} + \cdots + c_{q}(\theta). \tag{2.28}$$

Then, for any $(\theta, \lambda) \in f^{-1}(0) \cap (]a, b[\times\mathbb{R})$ there exist $\varepsilon, \delta > 0, n \in \mathbb{Z}^{+}$, and $h_{1}, \ldots, h_{n} \in H_{r}(\theta - \varepsilon, \theta + \varepsilon) with\ h_{1}(\theta) = \cdots = h_{n}(\theta) = \lambda$ such that

$$f^{-1}(0) \cap (]\theta - \varepsilon, \theta + \varepsilon[\times]\lambda - \delta, \lambda + \delta[) = \bigcup_{i=1}^{n} \Gamma(h_{i}). \tag{2.29}$$

Proof Let

$$(\theta, \lambda) \in f^{-1}(0) \cap (]a, b] \times \mathbb{R}). \tag{2.30}$$

By considering a change of variables, we may assume that $f^{-1}(0) \cap (]a,\ b[\ \times\ \mathbb{R})$ contains (0, 0) and that

$$(\theta, \lambda) = (0, 0) \tag{2.31}$$

without loss of generality.

By Proposition 2.3, there exist an open connected set $\hat{I}$ of $\mathbb{C}$ with $\hat{I} \supset I$ and an analytic function $\hat{f} : \hat{I} \times \mathbb{C} \to \mathbb{C}$ such that

$$f^{-1}(0) = \hat{f}^{-1}(0) \cap (I \times \mathbb{C}) = \hat{f}^{-1}(0) \cap (I \times \mathbb{R}). \tag{2.32}$$

Note that (2.32) implies that $\hat{f}(0, 0) = 0$, and $\hat{f}(0, \lambda) \not\equiv 0$. Thus, we see that $\hat{f}$ can be expressed in a neighborhood of (0, 0) by an element ψ of $\mathbb{C}\{z_1, z_2\}$ with $\psi(0, 0) = 0$ and $\psi(0, \lambda) \not\equiv 0$. (Recall that $\Pi(\mathbb{C}\{z_1, z_2\}) := \{\psi \in \mathbb{C}\{z_1, z_2\} : \psi(0, 0) = 0, \psi(0, \lambda) \not\equiv 0\}$). Namely, we see that there exist $\psi \in \Pi(\mathbb{C}\{z_1,\ z_2\})$ and $\boldsymbol{r} = (r_1, r_2) \in \nabla(\psi)$ such that

$$\hat{f}(\theta, \lambda) = \psi(\theta, \lambda) \tag{2.33}$$

for all $(\theta, \lambda) \in \Delta(\boldsymbol{r}) := \Delta(r_1) \times \Delta(r_2)$.

Fix such ψ and $\boldsymbol{r} = (r_1, r_2)$.

Let

$$W := \hat{f}^{-1}(0) \cap \Delta(\boldsymbol{r}), \tag{2.34}$$

which by (2.33) can be rewritten as the set of zeros of ψ in $\Delta(\boldsymbol{r})$:

$$W = \{(\theta, \lambda) \in \Delta(\boldsymbol{r}) : \psi(\theta, \lambda) = 0\}. \tag{2.35}$$

Now recall Proposition 3.5.C (Intermediate Target Proposition) in part IV [1], which implies that there exist

(i) $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^{+2}$,
(ii) $s \in \mathbb{R}^+$,
(iii) $n \in \mathbb{Z}^+, k_1, \ldots, k_n \in \mathbb{Z}^+$,
(iv) $h_1 \in H_0(\Delta(s^{1/k_1})), \ldots, h_n \in H_0(\Delta(s^{1/k_n}))$ with $h_1(0) = \cdots = h_n(0) = 0$,

such that the following equality holds:

$$W \cap \Delta(\boldsymbol{t}) \cap (\Delta(s) \times \mathbb{C}) = \bigcup_{i=1}^{n} A_i. \tag{2.36}$$

Here

$$A_i := \bigcup_{j=1}^{k_i} \varGamma(h_i \circ \nu_{i,j}), \tag{2.37}$$

where $\nu_{i,j} : \Delta(s) \to \Delta(s^{1/k_i})$ is the function defined by $\nu_{i,j}(x) = \hat{\nu}_{k_i,j}(x)$, and where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_i\}$.

Fix such $\boldsymbol{t} = (t_1, t_2), s, n, k_1, \ldots, k_n$ and $h_1, \ldots, h_n$.

We need the following two Lemmas 2.1 and 2.2:

Lemma 2.1 *The notation and the assumptions being as above, let I_0 be any subset of $I = [a, b]$. Then, we have*

$$A_i \cap (I_0 \times (\mathbb{C} - \mathbb{R})) = \emptyset, \tag{2.38}$$

for all $i \in \{1, \ldots, n\}$.

Proof of Lemma 2.1 Let I_0 be any subset of I, then (2.32) implies that

$$\hat{f}^{-1}(0) \cap (I \times (\mathbb{C} - \mathbb{R})) = \emptyset. \tag{2.39}$$

Since $W \subset \hat{f}^{-1}(0)$ and $I_0 \subset I$, we then see that

$$W \cap (I_0 \times (\mathbb{C} - \mathbb{R})) = \emptyset. \tag{2.40}$$

But, by (2.36), we have for all $i \in \{1, \ldots, n\}$

$$A_i \subset W \tag{2.41}$$

hence

$$\begin{aligned} &A_i \cap (I_0 \times (\mathbb{C} - \mathbb{R})) \\ &\subset W \cap (I_0 \times (\mathbb{C} - \mathbb{R})) = \emptyset, \end{aligned} \tag{2.42}$$

which shows the conclusion of the lemma holds. □

Lemma 2.2 *The notation and the assumptions being as above, there exist $h_{01}, \ldots, h_{0n} \in H_r(-s, s)$ with $h_{01}(0) = \cdots = h_{0n}(0) = 0$ such that*

$$\varGamma\,(h_{0i}) = A_i \cap (]-s, s[\, \times \mathbb{R}) = A_i \cap (]-s, s[\, \times \mathbb{C}) \tag{2.43}$$

for all $i \in \{1, \ldots, n\}$.

Proof of Lemma 2.2 Bearing in mind the assumed fact that 0 is in the interior of $I = [a, b]$, we can use the above Lemma 2.1 and Proposition 4.4 in part III [2] to get the conclusion of the lemma. □

Now, (2.34), (2.36), and (2.43) imply that

$$\hat{f}^{-1}(0) \cap \Delta(\boldsymbol{r}) \cap \Delta(\boldsymbol{t}) \cap (]-\varepsilon, \varepsilon[\times\mathbb{R}) = \bigcup_{i=1}^{n} \Gamma(h_{0i}|]-\varepsilon, \varepsilon[) \tag{2.44}$$

for all $\varepsilon \in]0, s]$. On the other hand, we know that by Proposition 2.1(iii),

$$f^{-1}(0) = \hat{f}^{-1}(0) \cap (I \times \mathbb{R}). \tag{2.45}$$

Hence, considering the intersection of $(I \times \mathbb{R})$ and each side of (2.44), we obtain

$$f^{-1}(0) \cap \Delta(\boldsymbol{r}) \cap \Delta(\boldsymbol{t}) \cap (]-\varepsilon, \varepsilon[\times\mathbb{R}) = \bigcup_{i=1}^{n} \Gamma(h_{0i}|(]-\varepsilon, \varepsilon[\cap I)) \tag{2.46}$$

for all $\varepsilon \in]0, s]$. Select an $\varepsilon \in]0, s]$ such that

$$]-\varepsilon, \varepsilon[=]-\varepsilon, \varepsilon[\cap I, \tag{2.47}$$

$$\varepsilon < \min(r_1, t_1), \tag{2.48}$$

and set

$$\delta = \min(r_2, t_2). \tag{2.49}$$

Then, we have

$$f^{-1}(0) \cap (]-\varepsilon, \varepsilon[\times]-\delta, \delta[) = \bigcup_{i=1}^{n} \Gamma(h_{0i}|]-\varepsilon, \varepsilon[), \tag{2.50}$$

which shows that the conclusion of the proposition is true. □

3 The second generation Fukui project, matrix art, and the asymptotic linearity theorem extension conjecture (ALTEC)

The Matrix Art Program is a philosophical and methodical extension, from science towards art, of Fukui's approach and also of the Approach via the Aspect of Form and General Topology (cf. [13] and references therein) in the RST, which is the central unifying theory in the First and Second Generation Fukui project. Part of the Second Generation Fukui Project directly related to the Matrix Art is currently called Niagara project, since the idea of this project was born after visiting the Corning Glass Museum on the way to Niagara Falls, USA, during an international exchange program between Tsuyama National College of Technology (TNCT) Japan and Pennsylvania College of Technology USA.

Members from the Fukui Project Association who are involved in the Matrix Art and related programs have been using computer programs in MATLAB for a set of experiments in molecular networks and in computer graphic art, using fundamental methods of the RST and also referring to Problems I $\sim$ IV given in ref. [5]. The picture in Fig. 1 is one of the examples of Matrix Art. The picture shows the (θ, λ) energy surface and energy band curves of a carbon nanotube. (The nickname of this picture is "Cat's Cradle"). It is noteworthy that the mathematical techniques developed in the present series of papers can be applied to these and other energy band curves. The reader is referred to ref. [8] where concrete multi-graphs mentioned in Sect. 1 were given in analytical forms.

Remarks 1 Pattern recognition, global pattern identification, and pattern analysis of matrix data are powerful theoretical tools in the study of molecular networks and of properties of molecules having many identical moieties. In Tsuyama National College of Technology (TNCT), Japan, matrix data of many molecular networks have been visualized using MATLAB software, and the visualized matrix data have been used as a new interface between science and art. Special thanks are due to T. Fukuda, H. Ikeda, N. Tsutsui, N. Tadamasa, T. Miuchi, K. Haruna, T. Komoto, I. Kishimoto, and members of the Fukui Project Association who contributed to the Matrix Art Program. We also remark that Matrix Art and Challenging Problems II $\sim$ IV given in ref. [5] are closely related to the unifying approach to the problems of spectral symmetry (cf. [16–20]) via the existence and uniqueness theorems of spectral resolution [19,20].

It is noteworthy that Matrix Art which involves fractal structure with self-similarity (see e.g. Fig. 1 in ref. [5] and Fig. 2 in the present paper) has been utilized to prove the following conjecture (ALTEC), which was presented in article [21] (entitled "Open problem, Magic Mountain and Devil's Staircase swapping problems") and is important in view of the Fukui conjecture—a guiding conjecture in the RST.

Asymptotic Linearity Theorem Extension Conjecture (ALTEC $C(I)$ version) *The Asymptotic Linearity Theorem* (*ALT*) *can not be extended from AC*(*I*) *to C*(*I*), *where AC*(*I*) *denotes the functional space of all real valued absolutely continuous functions defined on closed interval I, and C*(*I*) *denotes the functional space of all real valued continuous functions defined on closed interval I.*

This conjecture (ALTEC) was first proved by the first author (S.A.) of the present article, and the second proof of this conjecture was recently obtained by him in a seminar called "Matrix Art Challenge Seminar" in Tsuyama National College of Technology (TNCT), in conjunction with a continuous function $\mathrm{MagicMt}_\pi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ and the following Matrix Art pictures of the function called 3D Magic-mountain (π) and 2D Magic-mountain (π). The scale of the function has been changed in the pictures. The graph of the function $\mathrm{MagicMt}_\pi$ has an intersting self similarity and the nickname of the function $\mathrm{MagicMt}_\pi$ is "Tsuyama-castle function" ("Tsuyama-jyo kansu" in Japanese).

Details of the function $\mathrm{MagicMt}_\pi$ and the proofs of the ALTEC shall be published elsewhere.

Remarks 2 Pictures of Magic-mountain(π) in Fig. 2 were first obtained in the Matrix Art Challenge Seminar in TNCT in parallel with the procedure of the above-mentioned Niagara Project, which is a special new part of the on-going international, interdisciplinary, and inter-generational Second Generation Fukui Project.

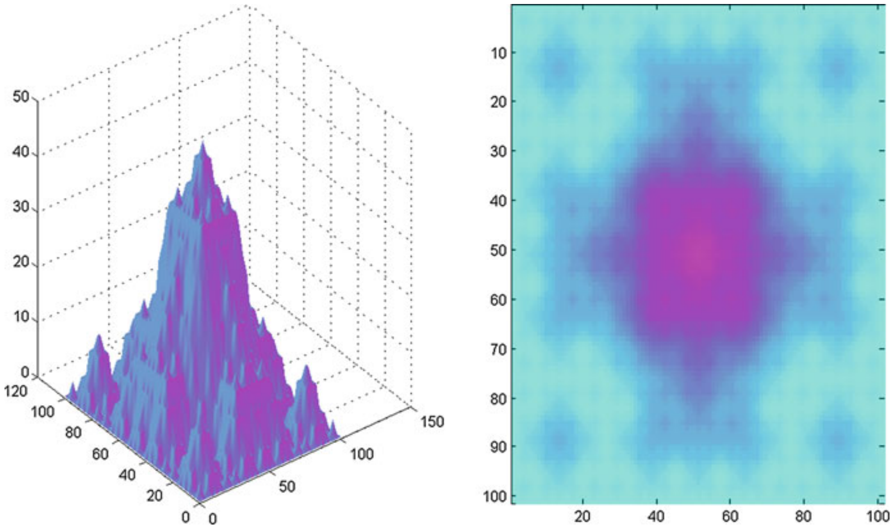

Fig. 2 3D magic-mountain (π) and 2D magic-mountain (π)

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